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# A Study of a Two Variables Gegenbauer Matrix Polynomials and Second Order Matrix Partial Differential Equations

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**Abstract**

Gegenbauer matrixes of two variable polynomials are introduced. the paper contains matrix differential recurrence relations, a matrix partial differential equation, double generating functions, new matrix of double and triple hypergeometric forms, a special property and a bilinear double generating function for the newly defined polynomial $C_{n,k}^{A}$ (x, y) and expansion of the Gegenbauer matrix polynomials as series of Hermite matrix polynomials.

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## 1. Introduction

*Defez and Jo'dar* $[4]$ introduced the elementary series manipulations for matrices $A(k,n)$ *and* $B(k,n)$ *in* $C^{N\times N}$. where $n \geq 0$, $k \geq 0$. For any matrix $P$ *in* $C^{N\times N}$ the exploit following relation define by due to $[10]$

$$(1-x)^{-P} = \sum_{n=0}^{\infty} \frac{(P)_n \, x^n}{n\,!} \quad , \quad |x| < 1 \tag{1}$$

and gives the hypergeometric matrix function $F(A,B;C;z)$ in the form

$$F(A,B;C;z)=\sum_{n=0}^{\infty}(A)_n(B)_n\left[(C)_n\right]^{-1}z^n \qquad (2)$$

*Batahan* [14], define the two- variable Hermite matrix polynomials by

$$H_n(x,y,A)=n!\sum_{n=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(x\sqrt{2A}\right)^{n-2k}y^k}{k!(n-2k)!}. \qquad (3)$$

$$x^n\left(\sqrt{2A}\right)^n=\sum_{k=0}^{[n/2]}\frac{n!y^k}{(n-2k)!k!}H_{n-2k}(x,y,A) \qquad (4)$$

*Sayyed ,Metwally and Batahan* [ 5 ] have defined and studied the Gegenbauer matrix polynomials by means of the relation

$$F=\left(1-2xt+t^2\right)^{-A}=\sum_{n=0}^{\infty}C_n^A(x)t^n \qquad (5)$$

where $A$ be appositive stable matrix in $C^{N\times N}$.
From (5), it follows that

$$C_n^A(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(A)_{n-k}}{k!(n-2k)!}(2x)^{n-2k} \qquad (6)$$

Clearly, $C_n^A(x)$ is a matrix polynomial of degree n in x.
*Khan and Khammash* [12 ], Define the Gegenbauer polynomials of two variables by

$$C_{n,k}^{\nu}(x,y)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^{r+j}(\nu)_{n+k-r-j}(2x)^{n-2r}\,2(2y)^{k-2j}}{r!\,j!\,(n-2r)!\,(k-2j)!} \qquad (7)$$

from which it follows that $C_{n,k}^{\nu}(x,y)$ is a polynomial in two variables x and y of degree precisely n in x and k in y. Thus $C_{n,k}^{\nu}(x,y)$ is a polynomial in two variables x and y of degree n + k. in this paper our main aim is to introduce and study, Gegenbauer matrix of two variable polynomials.

## 2. The Gegenbauer Matrix Polynomials of Two Variables

Let $A$ be matrix in $C^{N\times N}$ We define the Gegenbauer matrix polynomials of two variables denoted by $C_{n,k}^{A}(x,y)$ by the double generating relation

$$\left(1-2xs+s^2-2yt+t^2\right)^{-A}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}(x,y)s^nt^k \qquad (8)$$

By using the following series identity [ 4 ]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \qquad (9)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k) \qquad (10)$$

we have

$$\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A}=$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(A)_{n+k-r-j}\ (2\mathrm{x})^{\mathrm{n-2r}}\ (2\mathrm{y})^{\mathrm{k-2j}}\ (-1)^{\mathrm{r+j}}\ \mathrm{s}^{\mathrm{n}}\ \mathrm{t}^{\mathrm{k}}}{\mathrm{r}!\ \mathrm{j}!\ (\mathrm{n-2r})!\ (\mathrm{k-2j})!} \tag{11}$$

By equating the coefficients of $s^{n}t^{k}$ in $(8)$ and $(11)$, we obtain an explicit representation of the Gegenbauer matrix polynomials. In the form

$$C_{n,k}^{A}(\mathrm{x},\mathrm{y})=\sum_{\mathrm{r}=0}^{\left[\frac{\mathrm{n}}{2}\right]}\sum_{\mathrm{j}=0}^{\left[\frac{\mathrm{k}}{2}\right]}\frac{(-1)^{r+j}\ (A)_{n+k-r-j}\ (2\mathrm{x})^{n-2r}\ 2(2\mathrm{y})^{k-2j}}{\mathrm{r}!\ \mathrm{j}!\ (\mathrm{n-2r})!\ (\mathrm{k-2j})!} \tag{12}$$

From which it follows that $C_{n,k}^{A}(\mathrm{x},\mathrm{y})$ is a matrix polynomial in two variables x and y of degree precisely n in x and k in y. If in $(8)$, we replace x by –x and s by –s, the left side does not exchange, we obtain

$$C_{n,k}^{A}(-\mathrm{x},\mathrm{y})=(-1)^{n}\ C_{n,k}^{A}(\mathrm{x},\mathrm{y}) \tag{13}$$

Similarly, by replacing y by –y and t by –t in $(8)$ ,we obtain .

$$C_{n,k}^{A}(\mathrm{x},-\mathrm{y})=(-1)^{n}\ C_{n,k}^{A}(\mathrm{x},\mathrm{y}) \tag{14}$$

For x=y=1, w obtain

$$\left(1-2(s+t)+\left(s^{2}+t^{2}\right)\right)^{-A}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}(1,1)\mathrm{s}^{\mathrm{n}}t^{k}$$

so that $C_{n,k}^{A}(\mathrm{x},\mathrm{y})$ is an odd function of x for n odd, an even function of x for n even. Similarly $C_{n,k}^{A}(\mathrm{x},\mathrm{y})$ is an odd function of y for k odd, an even function of y for k, even. Similarly by replacing y by -y and t by -t in$(8)$ ,we get

$$C_{n,k}^{A}(-\mathrm{x},-\mathrm{y})=(-1)^{n+k}\ C_{n,k}^{A}(\mathrm{x},\mathrm{y})$$

Putting t=0 and s=0 in $(8)$, we get

$$C_{n,0}^{A}(\mathrm{x},\mathrm{y})=C_{n}^{A}(\mathrm{x})$$

$$C_{0,k}^{A}(\mathrm{x},\mathrm{y})=C_{k}^{A}(\mathrm{y})$$

where $C_{n}^{A}(\mathrm{x})$ , $C_{k}^{A}(\mathrm{y})$ is the Gegenbauer matrix polynomial $[\,5\,]$.

For $t=0$ and $x=1$, we get

$$(1-s)^{2A}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}(1,y)s^{n}$$

$$C_{n,k}^{A}(1,y)=\frac{(2A)_{n}}{n!} \tag{15}$$

and, for $s=0$ and $y=1$, we get

$$C_{n,k}^{A}(x,1)=\frac{(2A)_{k}}{k!}\ . \tag{16}$$

Also by $(8)$, we get for $x=0,y=0$

$$\left(1+s^2+t^2\right)^{-A} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{A}\ (0,0)\ s^n\ t^k .$$

However,

$$\left(1+s^2+t^2\right)^{-A} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+k}\ (A)_{n+k}\ s^{2n}\ t^{2k}}{n\ !\ k\ !}$$

and hence

$$\left.\begin{aligned} &C_{2n+1,2k}^{A}\ (0,0) = 0,\ C_{2n,2k+1}^{A}\ (0,0) = 0,\ C_{2n+1,2k+1}^{A}\ (0,0) = 0,\\ &C_{2n,2k}^{A}\ (0,0) = \frac{(-1)^{n+k}\ (A)_{n+k}}{n\ !\ k\,!} \end{aligned}\right\} \qquad (17)$$

The explicit representation $(12)$ gives

$$C_{n,k}^{\nu}\ (x,y) = \frac{2^{n+k}\ (\nu)_{n+k}\ x^n\ y^k}{n\ !\ k\ !} + \Pi_{n+k-2}$$

where $\Pi_{n+k-2}$ is matrix polynomial of degree $(n+k-2)$ in x and y

then it follows that

$$\frac{\partial^n}{\partial x^n} C_{n,k}^{A}\ (x\ ,y) = \frac{2^{n+k}\ (A)_{n+k}}{k\ !}\ y^k$$

$$\frac{\partial^k}{\partial x^k} C_{n,k}^{A}\ (x\ ,y) = \frac{2^{n+k}\ (A)_{n+k}}{n!}\ x^n$$

equation$(12)$ yields

$$\frac{\partial}{\partial x}\ C_{n,k}^{A}\ (x,y) = \sum_{r=0}^{\left[\frac{n-1}{2}\right]}\ \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^{r+j}\ (A)_{n+k-r-j}\ (2x)^{n-1-2r}\ (2y)^{k-2j}}{r\ !\ j\ !\ (n-1-2r)\ !\ (k-2j)\ !}$$

$$\frac{\partial}{\partial y}\ C_{n,k}^{A}\ (x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\ \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \frac{(-1)^{r+j}\ (A)_{n+k-r-j}\ (2x)^{n-2r}\ (2y)^{k-1-2j}}{r\ !\ j\ !\ (n-2r)\ !\ (k-1-2j)\ !}$$

$$\left.\begin{aligned}
\left[\frac{\partial}{\partial x} C^{A}_{2n+1,2k}(x,y)\right]_{x=0,y=0} &= \frac{(-1)^{n+k}\, 2(A)_{n+k+1}}{n!\, k!},\\
\left[\frac{\partial}{\partial x} C^{A}_{2n,2k}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial x} C^{A}_{2n,2k+1}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial x} C^{A}_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} &= 0,
\end{aligned}\right\} \tag{18}$$

Similarly,

$$\left.\begin{aligned}
\left[\frac{\partial}{\partial y} C^{A}_{2n,2k}(x,y)\right]_{x=0,y=0} &= 0\\
\left[\frac{\partial}{\partial y} C^{A}_{2n+1,2k}(x,y)\right]_{x=0,y=0} &= 0\\
\left[\frac{\partial}{\partial y} C^{A}_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} &= 0\\
\left[\frac{\partial}{\partial y} C^{A}_{2n,2k+1}(x,y)\right]_{x=0,y=0} &= \frac{(-1)^{n+k}\, 2(A)_{n+k+1}}{n!\, k!}
\end{aligned}\right\} \tag{19}$$

## 3. Matrix Differential Recurrence Relations

Differentiating the generating relation $(8)$

$$G = \left(1-2xs+s^2-2yt+t^2\right)^{-\nu} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}(x,y)\, s^n t^k \tag{20}$$

with respect to x and s yields , respectively ,we get

$$\frac{\partial G}{\partial x} = \frac{2s}{\left(1-2xs+s^2-2yt+t^2\right)} AG$$

$$\frac{\partial G}{\partial s} = \frac{2(x-s)}{\left(1-2xs+s^2-2yt+t^2\right)} AG \tag{21}$$

so that matrix function G satisfies the partial matrix differential equations .

$$(x-s)\frac{\partial G}{\partial x} - s\frac{\partial f}{\partial s} = 0$$

similarly, By differentiating $(20)$ with respect to y and t yields , respectively

$$\frac{\partial G}{\partial y} = \frac{2t}{\left(1-2xs+s^2-2yt+t^2\right)} AG$$

$$\frac{\partial G}{\partial t}=\frac{2\left(y-t\right)}{\left(1-2xs+s^{2}-2yt+t^{2}\right)}AG \qquad (22)$$

so that matrix function G satisfies the partial matrix differential equations .

$$\left(y-t\right)\frac{\partial G}{\partial y}-t\frac{\partial f}{\partial t}=0$$

with implies the matrix differential recurrence relations.

$$x\frac{\partial}{\partial x}C_{n,k}^{A}\left(\mathrm{x,y}\right)-nC_{n,k}^{A}\left(\mathrm{x,y}\right)=\frac{\partial}{\partial x}C_{n-1,k}^{A}\left(\mathrm{x,y}\right) \qquad (23)$$

Similarly differentiating $(20)$ with respected to y and t, yields

$$y\frac{\partial}{\partial y}C_{n,k}^{A}\left(\mathrm{x,y}\right)-kC_{n,k}^{A}\left(\mathrm{x,y}\right)=\frac{\partial}{\partial y}C_{n,k-1}^{A}\left(\mathrm{x,y}\right) \qquad (24)$$

and

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C_{n,k}^{A}\left(\mathrm{x,y}\right)-\left(n+k\right)C_{n,k}^{A}\left(\mathrm{x,y}\right)=\frac{\partial}{\partial x}C_{n-1,k}^{A}\left(\mathrm{x,y}\right)+\frac{\partial}{\partial y}C_{n,k-1}^{A}\left(\mathrm{x,y}\right) \qquad (25)$$

From $(20)$ with the aid of $(21)$ & $(22)$ ,we get respectively the following

$$2A\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k-1} \qquad (26)$$

$$2A\left(x-s\right)\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}nC_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n-1}t^{k} \qquad (27)$$

Since $1-s^2-t^2-2s(x-s)-2t(y-t)=1-2sx+s^2-2yt+t^2$, we may Multiply $(26)$ by $\left(1-s^{2}\right)$ and $(27)$ by -2s and subtracting $(27)$ from $(26)$ we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n-1}t^{k}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n+1}t^{k}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k+1}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}2nC_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}2kC_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k}=2A\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k}$$

we thus obtain another matrix differential recurrence relation

$$\left(2\left(n+k\right)I+2A\right)C_{n,k}^{A}\left(\mathrm{x,y}\right)=\frac{\partial}{\partial x}C_{n+1,k}^{A}\left(\mathrm{x,y}\right)-\frac{\partial}{\partial x}C_{n-1,k}^{A}\left(\mathrm{x,y}\right)-\frac{\partial}{\partial y}C_{n,k-1}^{A}\left(\mathrm{x,y}\right) \qquad (28)$$

From $(20)$ with the aid of $(22)$ we get respectively the following

$$2A\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k-1} \qquad (29)$$

$$2A\left(y-t\right)\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}kC_{n,k}^{A}\left(\mathrm{x,y}\right)s^{n}t^{k-1} \qquad (30)$$

with multiplying $(22-a)$ and $(23-a)$ by $-t^{2}$ and $-2t$ and adding $(23-a)$ to $(22-a)$ We obtain the another matrix differential recurrence relation

$$\left(2(n+k)I+2A\right)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial x}C_{n,k+1}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial x}C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial y}C_{n,k-1}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (31)$$

Adding $(28)$ successively to $(23)$, $(24)$ and $(25)$, we get

$$x\frac{\partial}{\partial x}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial x}C_{n+1,k}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial y}C_{n,k-1}^{A}\,(\mathrm{x},\mathrm{y})-(n+2k+2a)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (32)$$

$$y\frac{\partial}{\partial y}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial x}C_{n+1,k}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial x}C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y})-(2n+k+2A)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (33)$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial x}C_{n+1,k}^{A}\,(\mathrm{x},\mathrm{y})-(n+k+2\nu)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (34)$$

Adding $(31)$ successively to $(23)$, $(24)$ and $(25)$, we get

$$x\frac{\partial}{\partial x}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial y}C_{n,k+1}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial y}C_{n,k-1}^{A}\,(\mathrm{x},\mathrm{y})-(n+2k+2A)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (35)$$

$$y\frac{\partial}{\partial y}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial y}C_{n,k+1}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial x}C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y})-(2n+k+2A)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (36)$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=\frac{\partial}{\partial y}C_{n,k+1}^{A}\,(\mathrm{x},\mathrm{y})-(n+k+2A)C_{n,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (37)$$

Shifting the index from n to n – 1 in $(32)$ and using $(23)$, we get

$$\left(x^{2}-1\right)\frac{\partial}{\partial x}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=nxC_{n,k}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial y}C_{n-1,k-1}^{A}(\mathrm{x},\mathrm{y})-(n+2k+2A-I)C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y}) \qquad (38)$$

Similarly shifting the index from k to k – 1 in $(25-d)$ and using $(21-a)$, we get

$$\left(y^{2}-1\right)\frac{\partial}{\partial y}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})=kyC_{n,k}^{A}\,(\mathrm{x},\mathrm{y})-\frac{\partial}{\partial x}C_{n-1,k-1}^{A}(\mathrm{x},\mathrm{y})-(2n+k+2A-1)C_{n,k-1}^{A}(\mathrm{x},\mathrm{y}) \qquad (39)$$

## 4. Matrix of Partial differential Equation of $C_{n,k}^{A}$ (x, y)

From $(23)$ and $(24)$ we have

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\,C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y})=x\,\frac{\partial}{\partial x}\,C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})-nC_{n,k}^{A}\,(\mathrm{x},\mathrm{y})\\ &\frac{\partial^{2}}{\partial x^{2}}\,C_{n-1,k}^{A}\,(\mathrm{x},\mathrm{y})=x\,\frac{\partial^{2}}{\partial x^{2}}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})+(1-n)\frac{\partial}{\partial x}C_{n,k}^{A}\,(\mathrm{x},\mathrm{y})\end{aligned}\right\} \qquad (40)$$

$$\left.\begin{array}{l}\dfrac{\partial}{\partial y}C^{A}_{n,k-1}(\mathrm{x,\ y})=y\dfrac{\partial}{\partial y}C^{A}_{n,k}(\mathrm{x,\ y})-kC^{A}_{n,k}(\mathrm{x,\ y})\\[2ex] \dfrac{\partial^2}{\partial y^2}C^{A}_{n,k-1}(\mathrm{x,\ y})=y\dfrac{\partial^2}{\partial y^2}C^{A}_{n,k}(\mathrm{x,\ y})+(1-k)\dfrac{\partial}{\partial y}C^{A}_{n,k}(\mathrm{x,\ y})\end{array}\right\}\quad(41)$$

Shifting the index from n to n –1 in $(32)$ and from k to k – 1 in $(36)$ we get

$$x\frac{\partial}{\partial x}C^{A}_{n-1,k}(\mathrm{x,y})=\frac{\partial}{\partial x}C^{A}_{n,k}(\mathrm{x,y})-(n+2k+2A-I)C^{A}_{n-1,k}(\mathrm{x,y})-\frac{\partial}{\partial y}C^{A}_{n-1,k-1}(\mathrm{x,y})$$

$$y\frac{\partial}{\partial y}C^{A}_{n,k-1}(\mathrm{x,y})=\frac{\partial}{\partial y}C^{A}_{n,k}(\mathrm{x,y})-(n+2k+2A-I)C^{A}_{n,k-1}(\mathrm{x,y})-\frac{\partial}{\partial x}C^{A}_{n-1,k-1}(\mathrm{x,y})$$

Differentiate with respect to x and y respectively to find

$$x\frac{\partial^2}{\partial x^2}C^{A}_{n-1,k}(\mathrm{x,y})=\frac{\partial^2}{\partial x^2}C^{A}_{n,k}(\mathrm{x,y})-(n+2k+2A)\frac{\partial}{\partial x}C^{A}_{n-1,k}(\mathrm{x,y})-\frac{\partial^2}{\partial x\,\partial y}C^{A}_{n-1,k-1}(\mathrm{x,y})\qquad(42)$$

$$y\frac{\partial^2}{\partial y^2}C^{A}_{n,k-1}(\mathrm{x,y})=\frac{\partial^2}{\partial y^2}C^{A}_{n,k}(\mathrm{x,y})-(2n+k+2A)\frac{\partial}{\partial y}C^{A}_{n,k-1}(\mathrm{x,y})-\frac{\partial^2}{\partial x\partial y}C^{A}_{n-1,k-1}(\mathrm{x,y})\qquad(43)$$

From $(40)$ butting $\frac{\partial}{\partial x}C^{A}_{n-1,k}(x,y)$ and $\frac{\partial}{\partial x}C^{A}_{n-1,k}(x,y)$ into $(42)$ we get

$$(1-x^2)\frac{\partial^2}{\partial x^2}C^{A}_{n,k}(\mathrm{x,y})-(2k+2A+I)x\frac{\partial}{\partial x}C^{A}_{n,k}(\mathrm{x,y})+n(n+2k+2A)C^{A}_{n,k}(\mathrm{x,y})-\frac{\partial^2}{\partial x\,\partial y}C^{A}_{n-1,k-1}(\mathrm{x,y})=0\qquad(44)$$

Also, from $(41)$ putting $\frac{\partial}{\partial y}C^{A}_{n,k-1}(x,y)$ and $\frac{\partial}{\partial y}C^{A}_{n,k-1}(x,y)$ into $(43)$ we get

$$(1-y^2)\frac{\partial^2}{\partial y^2}C^{A}_{n,k}(\mathrm{x,y})-(2n+2A+I)y\frac{\partial}{\partial y}C^{A}_{n,k}(\mathrm{x,y})+k(2n+k+2A)C^{A}_{n,k}(\mathrm{x,y})-\frac{\partial^2}{\partial x\,\partial y}C^{A}_{n-1,k-1}(\mathrm{x,y})=0\qquad(45)$$

Subtracting $(45)$ from $(44)$,

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2}-(1-y^2)\frac{\partial^2}{\partial y^2}\right\}C^{A}_{n,k}(\mathrm{x,y})-\left\{(2(k+A)+1)x\frac{\partial}{\partial x}-(2n+2A+I)y\frac{\partial}{\partial y}\right\}C^{A}_{n,k}(\mathrm{x,y})$$

$$(n-k)((n+k)I+2A)C^{A}_{n,k}(\mathrm{x,\ y})=0\qquad(46)$$

we obtain the Gegenbauer's matrix of the partial differential equation differential equation satisfied by $C^{A}_{n,k}(\mathrm{x,\ y})$.

## 5. Hypergeometric matrix representations of $\mathbf{C^{\nu}_{n,k}(x,y)}$

Since

$$\left(1-2xs+s^2-2yt+t^2\right)^{-A}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{A}_{n,k}(x,y)\,s^n t^k$$

we note that

$$\left(1-2xs+s^2-2yt+t^2\right)^{-A}=$$

$$(1-s-t)^{-2A}\left[1-\frac{2s(x-1)}{(1-s-t)^2}-\frac{2t(y-1)}{(1-s-t)^2}-\frac{2st}{(1-s-t)^2}\right]^{-A} \qquad (47)$$

and from $[5]$, we have

$$(A)_{n+k}=(A)_n\,(A+nI)_K\,. \qquad (48)$$

So $(47)$ and $(48)$, permits us to write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}(x,y)s^nt^k$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)}\frac{(2A)_{\mathrm{n+k}}\left(2A+(\mathrm{n+k})I\right)_{j+p}(-nI)_{j+r}(-kI)_{p+r}(1-x)^j(1-y)^p s^nt^k}{2^{j+p+r}\left(\frac{2A+\mathrm{I}}{2}\right)_{j+p+r}j\,!\,\mathrm{p}!\,\mathrm{r}!\,\mathrm{n}!\,\mathrm{k}!}$$

Therefore

$$C_{n,k}^{A}(x,y)=\frac{(2A)_{\mathrm{n+k}}}{\mathrm{n}!\,\mathrm{k}!}\,\mathrm{F}^{(3)}\left[\begin{array}{l}-::-nI;-kI;2A+(n+k)I:-;-;-;\\ \qquad\qquad\qquad\qquad\frac{1-x}{2},\ \frac{1-y}{2},\frac{1}{2}\\ \frac{2A+I}{2}::-;-;-:-;-;-;\end{array}\right] \qquad (49)$$

Since $C_{n,k}^{A}$ $(-\mathrm{x},\ -\mathrm{y})$ = $(-1)^{\mathrm{n+k}}$ $C_{n,k}^{A}(-x,-y)=(-1)^{n+k}C_{n,k}^{A}(x,y)$, it follows from $(49)$ that also

$$C_{n,k}^{A}(x,y)=\frac{(-1)^{n+k}(2A)_{n+k}}{\mathrm{n}!\,\mathrm{k}!}\,\mathrm{F}^{(3)}\left[\begin{array}{l}-::-nI;-kI;2A+(n+k)I:-;-;-;\\ \qquad\qquad\qquad\qquad\frac{1+x}{2},\ \frac{1+y}{2},\frac{1}{2}\\ \frac{2A+I}{2}::-;-;-:-;-;-;\end{array}\right] \qquad (50)$$

Next, consider $(12)$ again

$$C_{n,k}^{A}(x,y)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^{r+j}\,(A)_{\mathrm{n+k-r-j}}\,(2\mathrm{x})^{n-2r}(2\mathrm{y})^{k-2j}}{\mathrm{r}!\,\mathrm{j}!\,(\mathrm{n}-2\mathrm{r})!\,(\mathrm{k}-2j)!}$$

we may write it as

$$C_{n,k}^{A}(x,y)=\frac{2^{n+k}(A)_{n+k}x^ny^k}{n\,!\,\mathrm{k}!}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{\left(-\frac{n}{2}I\right)_{\mathrm{r}}\left(\left(-\frac{n}{2}+\frac{1}{2}\right)I\right)_{\mathrm{r}}\left(-\frac{k}{2}I\right)_{\mathrm{j}}\left(\left(-\frac{k}{2}+\frac{1}{2}\right)I\right)_{\mathrm{j}}(-1)^{r+s}}{\mathrm{r}!\,\mathrm{j}!\,\left(A-(\mathrm{n}+k)I\right)_{\mathrm{r+j}}\,\mathrm{x}^{2\mathrm{r}}y^{2j}}$$

or in terms of Kampe de Feriet’s double hypergeometric function, we have

$$C^{A}_{n,k}=\frac{2^{n+k}\,(A)_{n+k}\,x^{n}y^{k}}{n!\,k!}\,F\left[\begin{matrix}-:-\frac{n}{2}I,\left(-\frac{n}{2}+\frac{1}{2}\right)I;-\frac{k}{2}I,\left(-\frac{k}{2}+\frac{1}{2}\right)I; & \\ & -\frac{1}{x^{2}},\ -\frac{1}{y^{2}} \\ A-(n+k)I\ :\ -;\ -\ ; & \end{matrix}\right] \tag{51}$$

## 6. Additional double Generating Functions

The generating function $(1-2xs+s^{2}-2yt+t^{2})^{-A}$ used to define a polynomial $C^{A}_{n,k}(x,y)$ in two variables x and y analogue to Gegenbauer polynomials $C^{A}_{n}(x)$ in a single variable x can be expanded in powers of s and t in new ways, thus yielding additional results. For instance

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{A}_{n,k}(x,y)s^{n}t^{k}=\left(1-2xs+s^{2}-2yt+t^{2}\right)^{-A}$$

$$=(1-xs-yt)^{-2A}\left[1-\frac{s^{2}(x^{2}-1)}{(1-xs-yt)^{2}}-\frac{t^{2}(y^{2}-1)}{(1-xs-yt)^{2}}-\frac{2xyst}{(1-xs-yt)^{2}}\right]^{-A}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2A)_{n+k}\,(x^{2}-1)^{j}\,(y^{2}-1)^{p}\,s^{n}t^{k}x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2A+I}{2}\right)_{j+p+r}j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!}\,.$$

equating the coefficients of $s^{n}t^{k}$, we obtain

$$C^{\nu}_{n,k}(x,y)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2A)_{n+k}!\,(x^{2}-1)^{j}\,(y^{2}-1)^{p}\,x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2A+I}{2}\right)_{j+p+r}j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!} \tag{52}$$

More over, by exploiting $(52)$ and using the *Defez and Jo'dar* $[\,4\,]$, for the double sum. for arbitrary $c$ .

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}\,C^{A}_{n,k}(x,y)s^{n}t^{k}}{(2A)_{n+k}}$$

$$=\sum_{n,k=0}^{\infty}\frac{(c)_{n+k}\,s^{n}t^{k}}{(2A)_{n+k}}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2A)_{n+k}\,(x^{2}-1)^{j}\,(y^{2}-1)^{p}\,x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2A+I}{2}\right)_{j+p+r}j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!}$$

We will obtain the generating relation for the Gegenbauer matrix of two variable polynomials By identification of the coefficients of $s^{n}t^{k}$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}\,C^{A}_{n,k}(x,y)s^{n}t^{k}}{(2A)_{n+k}}$$

$$=\left(1-xs-yt\right)^{-c}\ F^{(3)}\left[\begin{array}{l} \frac{c}{2},\frac{c}{2}+\frac{1}{2}::-;-;-:-;-;-; \\ \qquad\qquad\qquad \frac{s^2\left(x^2-1\right)}{\left(1-xs-yt\right)^2},\frac{t^2\left(y^2-1\right)}{\left(1-xs-yt\right)^2},\frac{2xyst}{\left(1-xs-yt\right)^2} \\ \frac{2A+I}{2}::-;-;-:-;-;-; \end{array}\right] \tag{53}$$

Here $F^{(3)}[x, y, z]$ is a triple hypergeometric series [cf. Srivastava [16], p.428]. we have thus discovered the family of double generating functions in which c may be any complex number.

Let us now return to $(52)$ and consider the double sum

$$\sum_{n,k=0}^{\infty}\frac{C_{n,k}^{A}\left(\mathrm{x,y}\right)\,s^n t^k}{\left(2A\right)_{n+k}}$$

$$=\sum_{n,k=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{\left(x^2-1\right)^j\ \left(y^2-1\right)^p\ x^{n-2j}y^{k-2p}\ \mathrm{s}^n t^k}{2^{2j+2p+r}\left(\frac{2A+I}{2}\right)_{j+p+r}\ \mathrm{j!\,p!\,r!}\ \left(\mathrm{n}-2j-\mathrm{r}\right)!\ \left(\mathrm{k}-2p-\mathrm{r}\right)!}$$

We get,

$$=e^{xs+yt}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{j\,!\ \mathrm{p}\,!\ \mathrm{r}\,!\left(\frac{2A+I}{2}\right)_{j+p+r}}\left\{\frac{s^2\left(x^2-1\right)}{4}\right\}^j,\left\{\frac{t^2\left(y^2-1\right)}{4}\right\}^p\left\{\frac{xyst}{2}\right\}^r$$

$$\sum_{n,k=0}^{\infty}\frac{C_{n,k}^{A}\left(\mathrm{x,y}\right)\,s^n t^k}{\left(2A\right)_{n+k}}=e^{xs+yt}\ F^{(3)}\left[\begin{array}{l} -::-;-;-:-;-;-; \\ \qquad\qquad \frac{s^2\left(x^2-1\right)}{4},\frac{t^2\left(y^2-1\right)}{4},\frac{xyst}{2} \\ \frac{2A+I}{2}::-;-;-:-;-;-; \end{array}\right] \tag{54}$$

## 7. Special cases

a) Know from the definition $[8]$. We have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}\left(\mathrm{x,y}\right)s^n t^k=\rho^{-1} \tag{55}$$

where

$$\rho=\left(1-2xs+s^2-2yt+t^2\right)^A$$

In $(55)$, if we replace $x$ by $\frac{x-s}{\rho}$, $y$ by $\frac{y-t}{\rho}$, $s$ by $\frac{u}{\rho}$ and $t$ by $\frac{v}{\rho}$, we may write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{v}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)\rho^{-n-k-2A}u^n v^k$$

$$=\left[1-2x\left(s+u\right)+\left(s+u\right)^{2}-2y\left(t+v\right)+\left(t+v\right)^{2}\right]^{-A}$$

which by $(55)$ yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{a}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)\rho^{-n-k-2A}u^{n}v^{k}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}\left(x,y\right)\left(s+u\right)^{n}\left(t+v\right)^{k}$$

by equation the coefficients of $u^{n}v^{k}$ in the above , we get the special property of $C_{n,k}^{A}\left(x,y\right)$.

$$\rho^{-n-k-2A}C_{n,k}^{A}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(n+r\right)!\left(\mathrm{k}+\mathrm{j}\right)!C_{n+r,k+j}^{A}\left(\mathrm{x,y}\right)s^{r}\,\mathrm{t}^{\mathrm{j}}}{\mathrm{r}!\,j!\,\mathrm{n}!\,\mathrm{k}!} \qquad (56)$$

in which $\rho=\left(1-2xs+s^{2}-2yt+t^{2}\right)^{A}$ .

b) In the generating function $(54)$ by replacing $x$ by $\frac{x-s}{\rho}$ , $y$ by $\frac{y-t}{\rho}$, $s$ by $\frac{-u}{\rho}$ and $t$ by $\frac{-v}{\rho}$ and multiply each member by $\rho^{-2A}$

where $\rho=\left(1-2xs+s^{2}-2yt+t^{2}\right)^{A}$ and by using $(54)$, we get a bilinear double generating obtain

$$\rho^{-2A}\,e^{\left(-\frac{su\left(x-s\right)+tv\left(y-t\right)}{\rho^{2}}\right)}\times$$

$$F^{(3)}\left[\begin{array}{l} -::-;-;-:-;-;-; \\ \qquad\qquad\qquad \dfrac{u^{2}s^{2}\left(x^{2}-1+2yt-t^{2}\right)}{4\rho^{4}},\dfrac{v^{2}t^{2}\left(y^{2}-1+2xs-s^{2}\right)}{4\rho^{2}},\dfrac{uvst\left(x-s\right)\left(y-t\right)}{2\rho^{4}} \\ \dfrac{2A+I}{2}::-;-;-:-;-;-; \end{array}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{2}\left[-nI,-kI;2A;u,v\right]C_{n,k}^{A}\left(x,y\right)s^{n}t^{k}$$

where $\Phi_{2}$ is one of the seven confluent of the forms of the four Appell series defined by Humbert $\left[16,p.45\right]$.

## 8. Expand the Gegenbauer matrix polynomials of two variables in series of $H_{n}\left(x,y,A\right)$.

Let us now employ $(12)$ *and* $(4)$ and taking into account that each matrix commutes with it self. from $(12)$ ,one gets

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}\left(x,y\right)t^{n+k}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{j=0}^{[k/2]}\frac{(-1)^{r+j}(A)_{n+k-r-j}\,2^{n+k}x^{n-2r}y^{k-2r}t^{n+k}}{r!\,j!\,(n-2r)!\,(k-2j)!} \tag{57}$$

According to $[4]$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k) \tag{58}$$

where $A(k,n)$ is a matrix on $C^{N\times N}$. We have

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{r+j}(A)_{n+k+r+j}\,2^{n+k}x^{n}y^{k}t^{n+k+2r+2j}}{r!\,j!\,n!\,k!}$$

From $(4)$, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}(x,y)t^{n+k}\left(\sqrt{2A}\right)^{n}2^{-(n+k)}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{[n/2]}\frac{(-1)^{r+j}(A)_{n+k+r+j}\,n!\,y^{k}(y+1)^{s}t^{n+k+2r+2j}H_{n-2s}(x,y,A)}{r!\,j!\,n!\,k!\,(n-2s)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{k}\frac{(-1)^{r+j}\left[A+(n+k+r+j)I\right]_{S}y^{k-s}(y+1)^{s}t^{n+k+2r+2j+s}}{n!\,k!\,r!\,j!\,s!\,(k-s)!}$$

By using $(3)$ $[5,p103]$

$$\frac{(-1)^{k}}{(n-k)!}I=\frac{(-n)_{k}}{n!}I=\frac{(-nI)_{k}}{n!}\qquad,\qquad 0\le k\le n$$

it follows that

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{r+j}y^{k}}{n!\,k!\,r!\,j!}\sum_{s=0}^{k}\frac{\left[A+(n+k+r+j)I\right]_{S}(-kI)_{S}}{(-1)^{s}s!}$$
$$.\left(\frac{y+1}{y}\right)(A)_{n+k+r+j}H_{n}(x,y,A)t^{n+k+2r+2j}$$

and we may write as

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{r+j}y^{k}}{n!\,k!\,r!\,j!}\;{}_2F_0\left[-kI,A+(n+k+r+j)I;-;\left(-\frac{y+1}{y}t\right)\right]$$
$$.\;(A)_{n+k+r+j}H_{n}(x,y,A)t^{n+k+2r+2j}$$

Again from $(58)$, one gets

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{j=0}^{[k/2]}\frac{(-1)^{r+j}y^{k-2j}}{n!\,k!\,r!\,j!}\;{}_2F_0\left[-kI,A+(n+k+r+j)I;-;\left(-\frac{y+1}{y}t\right)\right]$$
$$.\;(A)_{n+k+r+j}H_{n}(x,y,A)t^{n+k}$$

Equating the coefficient of $t^{n+k}$ we obtain an expansion of the two-variable Gegenbauer matrix polynomials as series of two-variable Hermite matrix polynomials in the form

$$C_{n,k}^{A}(x,y) = \sum_{r=0}^{[n/2]} \sum_{j=0}^{[k/2]} \frac{(-1)^{r+j}\, y^{k-2j}}{!r!\,j!\,(n-2r)!\,(k-2j)!}$$

$$\cdot {}_2F_0\left[-kI, A+(n+k+r+j)I; -; \left(-\frac{y+1}{y}t\right)\right](A)_{n+k-r-j}\, H_n(x,y,A) \qquad (59)$$

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